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On Intrinsically Random $\mathbb{Z}^2$-Actions on a Lebesgue Space

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It is shown that a $\mathbb{Z}^2$-action on a Lebesgue space is intrinsically random (*IR*) iff it is a Kolmogorov action (*K*-action). As a consequence we obtain the fact that the $\mathbb{Z}^2$-action defined by the Lorentz gas is an *IR*-action and the $\mathbb{Z}^2$-action defined by the ideal gas is not an *IR*-action.

KEY WORDS: Intrinsical randomness; Kolmogorov $\mathbb{Z}^2$-actions.

1. INTRODUCTION

The concept of a weak (strong) intrinsically random $\mathbb{Z}^2$-action on a Lebesgue space has been introduced in ref. 2 and it is shown that any Kolmogorov (Bernoulli) $\mathbb{Z}^2$-action is intrinsically weak (strong) random.

An important role in the definition of an intrinsically random (weak and strong) action is played by a continuity condition (the condition (b) of Definition 2 in ref. 2) which has no corresponding analogue in the case of intrinsically random $\mathbb{Z}$-actions.

It was a natural question whether an intrinsically random (weak or strong) $\mathbb{Z}^2$-action is a *K*-action.

In this paper we first modify our definition of an *IWR*-action in a similar way as in ref. 4 (for $\mathbb{Z}$-actions). By this we replace the continuity condition by a more natural one. We call the actions obtained thereby as intrinsically random.

The main result of our paper says that the class of *IR*-actions coincides with the class of *K*-actions.

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Goldstein has shown in ref. 3 that the $\mathbb{Z}^2$-action defined by the Lorentz gas is a K-system and that the $\mathbb{Z}^2$-action defined by the ideal gas has zero entropy.

Thus from our result follows that the first $\mathbb{Z}^2$-action is an *IR*-action and the second is not an *IR*-action.

2. RESULT

Let $(X, \mathscr{B}, \mu)$ be a Lebesgue probability space and let $\mathscr{N}$ denote the trivial σ-algebra of X. Let $\mathscr{D} \subset L^2(X, \mu)$ stand for the set of all densities, i.e., measurable functions $f \geqslant 0$ with $\int_X f\, d\mu = 1$.

We denote by $\mathbb{Z}^2$ the group of two-dimensional integers. Let $\prec$ be the lexicographical order of $\mathbb{Z}^2$ and let $\Pi(N)$ stand for the set of positive (negative) vectors of $\mathbb{Z}^2$ with respect to $\prec$.

Let Φ be a $\mathbb{Z}^2$-action on $(X, \mathscr{B}, \mu)$, i.e., Φ is a homomorphism of $\mathbb{Z}^2$ into the group $\mathrm{Aut}(X, \mu)$ of all measure-preserving automorphisms of $(X, \mathscr{B}, \mu)$.

We denote by Φ^g the automorphism of $(X, \mathscr{B}, \mu)$, being the image of $g \in \mathbb{Z}^2$ under Φ.

Let $U = U_\Phi$ be the Koopman unitary representation of $\mathbb{Z}^2$ in $L^2(X, \mu)$ defined by the formula

$$U^g f = f \circ \Phi^g, \qquad f \in L^2(X, \mu), \qquad g \in \mathbb{Z}^2.$$

For a given σ-algebra $\mathscr{A} \subset \mathscr{B}$ and a function $f \in L^1(X, \mu)$, we denote by $E^{\mathscr{A}} f$ the conditional expectation of f given $\mathscr{A}$. In particular, we put

$$Ef = E^{\mathscr{N}} f = \int_X f\, d\mu.$$

We shall use in the sequel the following well known property of conditional expectations.

$$U_T \circ E^{\mathscr{A}} = E^{T^{-1}\mathscr{A}} \circ U_T \tag{1}$$

where $T \in \mathrm{Aut}(X, \mu)$ and U_T is the Koopman unitary operator associated with T.

For a given finite measurable partition ξ of X, we denote by $h(\xi, \Phi)$ the mean entropy of ξ with respect to Φ and by $h(\Phi)$ the entropy of Φ.[1]

Now we recall the definition of a K-action of $\mathbb{Z}^2$ on a Lebesgue space.[7]

An ordered pair (A, B) of subsets of $\mathbb{Z}^2$ is said to be *a cut* if they form a non-trivial partition of $\mathbb{Z}^2$ and for every $g \in A$ and $h \in B$ we have $g \prec h$.

A cut (A, B) is called *a gap* if A does not contain a greatest element and B does not contain a lowest element.

Definition 1. A $\mathbb{Z}^2$-action Φ is said to be a K-action if there exists a σ-algebra $\mathscr{A} \subset \mathscr{B}$ with

(i) $\Phi^g \mathscr{A} \subset \mathscr{A}$ for every $g \in N$,

(ii) the family $(\Phi^g \mathscr{A}, g \in \mathbb{Z}^2)$ is continuous, i.e., for every gap (A, B) of $\mathbb{Z}^2$ it holds

$$\bigvee_{g \in A} \Phi^g \mathscr{A} = \bigcap_{g \in B} \Phi^g \mathscr{A},$$

(iii) $\bigvee_{g \in \mathbb{Z}^2} \Phi^g \mathscr{A} = \mathscr{B}$,

(iv) $\bigcap_{g \in \mathbb{Z}^2} \Phi^g \mathscr{A} = \mathscr{N}$.

A σ-algebra $\mathscr{A}$ satisfying (i)–(iv) is called a K-σ-algebra.

It has been shown in ref. 7 that Φ is a K-action iff Φ has a completely positive entropy, i.e., $h(\xi, \Phi) > 0$ for every non-trivial finite measurable partition ξ of X and in ref. 6 that Φ is a K-action iff Φ is K-mixing.

Definition 1 can be rewritten in a more clear form, by the use of automorphisms $T = \Phi^{(1,0)}$ and $S = \Phi^{(0,1)}$ generating Φ, as follows

(i′) $S^{-1}\mathscr{A} \subset \mathscr{A}, T^{-1}\mathscr{A}_S \subset \mathscr{A}$ where $\mathscr{A}_S = \bigvee_{n=-\infty}^{\infty} S^n \mathscr{A}$,

(ii′) $\bigcap_{n=-\infty}^{+\infty} S^n \mathscr{A} = T^{-1}\mathscr{A}_S$,

(iii′) $\bigvee_{n=-\infty}^{+\infty} T^n \mathscr{A}_S = \mathscr{B}$,

(iv′) $\bigcap_{n=-\infty}^{+\infty} T^n \mathscr{A}_S = \mathscr{N}$.

For any linear operator U of $L^2(X, \mu)$ we denote by $\ker U$ the kernel of U.

A linear operator W of $L^2(X, \mu)$ is said to be doubly stochastic if it is positive, $W1 = 1$ and $E \circ W = E$, i.e., if it is a Markov operator preserving μ.

For a function f defined on Π and taking values in some linear normed space we write

$$f(g) \to x \qquad \text{as} \quad g \to +\infty$$

if for any $\varepsilon > 0$ there exists $g_0 \in \Pi$ with $\|f(g) - x\| < \varepsilon$ for all $g \succ g_0$.

Definition 2. We say that a generalized sequence $(W^g, g \in \Pi)$ of linear operators of $L^2(X, \mu)$ tends to equilibrium if it is a semigroup, all operators W^g, $g \in \Pi$ are doubly stochastic, for any $f \in \mathscr{D}$ the sequence $(\|W^g f - 1\|, g \in \Pi)$ is nonincreasing and $W^g f \to 1$ as $g \to +\infty$.

Definition 3. A $\mathbb{Z}^2$-action Φ is called an *IR*-action if there exists an operator P of $L^2(X, \mu)$ and a sequence $(W^g, g \in \Pi)$ tending to equilibrium such that

(a) P is a positive projection,

(b) $P1 = 1$,

(c) $\bigcap_{g \in \mathbb{Z}^2} \ker P \circ U^g = \{0\}$,

(d) for any $f \in L^2(X, \mu)$ and $m \in \mathbb{Z}$ we have

$$\lim_{n \to \infty} \|U^{(m,n)} \circ W^{(m,n)} f - U^{(m+1,-n)} \circ W^{(m+1,-n)} f\| = 0,$$

(e) for any $g \in \Pi$ the following diagram commutes:

$$\begin{array}{ccc} L^2(X,\mu) & \xrightarrow{P} & L^2(X,\mu) \\ \downarrow{\scriptstyle U^{-g}} & & \downarrow{\scriptstyle W^g} \\ L^2(X,\mu) & \xrightarrow[P]{} & L^2(X,\mu). \end{array}$$

It is easy to check that if Φ is an *IR*-action then it is intrinsically weak random in the sense od Definition 2 from ref. 2.

Remark 1. It is easy to see that the property of the intrinsical randomness is preserved if we pass from the ordered basis (e_1, e_2), $e_1 = (1, 0)$, $e_2 = (0, 1)$ to another one (e'_1, e'_2) by an automorphism A of $\mathbb{Z}^2$ which preserves the lexicographical order, i.e., the associated matrix of which is of the form $\begin{pmatrix} 1 & 0 \\ r & 1 \end{pmatrix}$ where $r \in \mathbb{Z}$.

Theorem. A $\mathbb{Z}^2$-action Φ is an *IR*-action iff it is a K-action.

Proof. *Necessity.* It has been shown in ref. 4 that (a) and (b) imply that P is strictly positive and so, by Corollary 3.4 to the representation theorem from ref. 5, the operator P is in fact a conditional expectation operator with respect to some σ-algebra $\mathscr{A}$, i.e., $P = E^{\mathscr{A}}$.

We claim that $\mathscr{A}$ is a K-σ-algebra for Φ.

Let $\mathscr{A}_g = \Phi^g \mathscr{A}$ and let $P^g = E^{\mathscr{A}_g}$, $g \in \mathbb{Z}^2$.

Applying the equality (1) it is easy to see that

$$P^g = U^{-g} \circ P \circ U^g, \qquad g \in \mathbb{Z}^2. \tag{2}$$

In order to check (i) it is enough to show that $P^g \circ P = P^g$ for any $g \in N$.

It follows from (2) and (e) that

$$P^g \circ P = U^{-g} \circ P \circ U^g \circ P = U^{-g} \circ W^{-g} \circ P^2$$
$$= U^{-g} \circ W^{-g} \circ P = U^{-g} \circ P \circ U^g = P^g.$$

Now we check (ii′) instead of (ii). Let $f \in L^2(X, \mu)$ be $\bigcap_{n=0}^{\infty} S^{-n}\mathscr{A}$-measurable. Hence it is of course $\mathscr{A}$-measurable, i.e., $Pf = f$.

It follows from (e) and (2) that

$$\|U^{(0,n)} \circ W^{(0,n)} f - U^{(1,-n)} \circ W^{(1,-n)} f\|$$
$$= \|U^{(0,n)} \circ W^{(0,n)} \circ Pf - U^{(1,-n)} \circ W^{(1,-n)} \circ Pf\|$$
$$= \|U^{(0,n)} \circ P \circ U^{-(0,n)} f - U^{(1,-n)} \circ P \circ U^{-(1,-n)} f\|$$
$$= \|P^{-(0,n)} f - P^{-(1,-n)} f\|$$
$$= \|E^{S^{-n}\mathscr{A}} f - E^{T^{-1}S^n\mathscr{A}} f\|, \qquad n \geqslant 1.$$

Hence taking the limit as $n \to \infty$ and applying (d) and the Doob martingale convergence theorem we get

$$E^{\bigcap_{n=0}^{\infty} S^{-n}\mathscr{A}} f = E^{T^{-1}\mathscr{A}_S} f.$$

It follows from our assumption that the left hand side is equal to f and so

$$f = E^{T^{-1}\mathscr{A}_S} f,$$

i.e., f is $T^{-1}\mathscr{A}_S$-measurable. This means that (ii′) and so (ii) is satisfied.

In order to prove (iii) we put

$$\mathscr{A}_\infty = \bigvee_{g \in \mathbb{Z}^2} \mathscr{A}_g.$$

Let $f \in L^2(X, \mu) \ominus L^2(X, \mathscr{A}_\infty, \mu)$, i.e., f is orthogonal to all subspaces $L^2(X, \mathscr{A}_g, \mu)$, $g \in \mathbb{Z}^2$.

Let $g \in \mathbb{Z}^2$ be arbitrary. Thus for every $\varphi \in L^2(X, \mu)$ we have

$$0 = (f, P^g \varphi) = (P^g f, \varphi).$$

Therefore $P^g f = 0$ and so (2) gives

$$U^{-g} \circ P \circ U^g f = 0, \qquad g \in \mathbb{Z}^2.$$

Hence

$$P \circ U^g f = 0, \qquad g \in \mathbb{Z}^2$$

and thus (c) implies $f = 0$.

This means that

$$L^2(X, \mu) = L^2(X, \mathscr{A}_\infty, \mu),$$

i.e., (iii) is true.

Let $\mathscr{A}_{-\infty} = \bigcap_{g \in \mathbb{Z}^2} \mathscr{A}_g$ and let $f \in L^2(X, \mu)$ be $\mathscr{A}_{-\infty}$-measurable. In order to check (iv) it is enough to show that f is constant.

Since $f = \max(f, 0) - \max(-f, 0)$ we may assume $f \geqslant 0$. If $Ef = 0$ then the proof is finished. If not then $\varphi = \frac{f}{Ef} \in \mathscr{D}$ and since $(W^g, g \in \Pi)$ tends to equilibrium we have

$$\|W^g \varphi - 1\| \to 0 \qquad \text{as} \quad g \to +\infty. \tag{3}$$

Let $g \in \Pi$. Since $\mathscr{A}_{-\infty} \subset \mathscr{A}_{-g}$ and φ is $\mathscr{A}_{-\infty}$-measurable we have $P^{-g}\varphi = \varphi$.

Hence

$$W^g \varphi = W^g \circ P\varphi = P \circ U^{-g} \circ P\varphi = P \circ U^{-g}\varphi = U^{-g} \circ P^{-g}\varphi = U^{-g}\varphi$$

and so (3) implies

$$\|\varphi - 1\| = \|U^{-g}\varphi - 1\| \to 0 \qquad \text{as} \quad g \to +\infty.$$

Therefore $\varphi = 1$ and so f is constant.

Sufficiency. Let Φ be a K-action of $\mathbb{Z}^2$ on $(X, \mathscr{B}, \mu)$ and let $\mathscr{A}$ be a corresponding K-σ-algebra.

In order to show that Φ is an *IR*-action we consider $P = E^{\mathscr{A}}$ and $W^g = P \circ U^{-g}$, $g \in \Pi$. Since $\mathscr{A} \subset \Phi^g \mathscr{A}$, $g \in \Pi$ the equality (1) implies $\{W^g, g \in \Pi\}$ is a semigroup.

The properties (a) and (b) are obvious.

It has been checked in ref. 2 that (e) and (f) are also fulfilled where one should substitute $\Lambda = P$.

Now let us suppose $f \in \ker \bigcap_{g \in \mathbb{Z}^2} PU^g$. Hence applying (2) we have

$$P \circ U^g f = U^g \circ P^g f = U^g \circ E^{\mathscr{A}_g} f = 0$$

and therefore $E^{\mathscr{A}_g} f = 0$, $g \in \mathbb{Z}^2$. Now the property (iii) and the Doob martingale convergence theorem say that $f = 0$, i.e., (c) is satisfied.

In order to check (d) it is enough to assume that $m=0$.

For any $f \in L^2(X, \mu)$ we have by (1)

$$\begin{aligned}
&\|U^{(0,n)} \circ W^{(0,n)} f - U^{(1,-n)} \circ W^{(1,-n)} f\| \\
&\quad = \|U_S^n \circ E^{\mathscr{A}} \circ U_S^{-n} f - U_T \circ U_S^{-n} \circ E^{\mathscr{A}} \circ U_T^{-1} \circ U_S^n f\| \\
&\quad = \|E^{S^{-n}\mathscr{A}} f - E^{T^{-1}S^n\mathscr{A}} f\|, \qquad n \geqslant 1.
\end{aligned}$$

Applying (ii′) and the Doob theorem mentioned above we get (d).

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